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Welfare Maximizing Prices Under Uncertainty

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Abstract

This paper presents a new result in the theory of optimal pricing for public utilities or regulated monopolies. The derivation is based on a detailed model of consumers and suppliers which represents uncertainty and inter-temporal linking effects such as investment and storage. Thus the time evolution of the industry is accounted for. The optimal pricing structure would cause individual profit maximizing responses to be welfare maximizing. It contains two terms: Short Run Marginal Cost pricing as well as a new "incentive term" to account for the interaction of participants at different time points. A probabilistic forecast of pricing structures at future times is also required.

1. Introduction

The problem addressed in this paper is the setting of prices to cause an efficient allocation of resources. A theoretical analysis incorporating inter-temporal linkages such as investment, and the effects of uncertainty yields a new result for welfare maximizing prices.

The main application of these results is to price setting for public or regulated utilities such as those which supply electricity or gas. However, in order to represent substitution of the utility product, such as gas replacing electricity, a number of products are considered. These are referred to as the "industry products". Further, in order to obtain a (welfare maximizing) balance between "supply-side" and "demand-side" generation options, the industry model consists of a number of participants, each of whom, at each time point, can either nett consume or nett produce each of the industry products. A nett producer of a product at one time can be a nett consumer at another.

Many participants possess significant inter-temporal linkages. That is, an operation or investment decision taken at one time point can affect the state of the plant at future times and thus influence the decisions that can then be taken. Decisions concerning material or thermal storages, product or factor stockpiles and start-up and shut-down sequences are significant demand-side examples. On the supply side of an electricity industry, linking is caused by, inter alia, thermal generator start-up times, hydro-electric and pumped storage units, fuel purchase agreements and pollution constraints.

Investment is a special case of inter-temporal linking, where a decision taken at one time point will affect plant capacity or technical efficiency at some future time, possibly after a long lead time.

At the time of making an operational or investment decision, there are many phenomena which, although they may affect the future operation of a participant's plant, cannot be known exactly. For example, items of plant break down and are repaired, demand for the outputs of some manufacturing processes change with time and the future of both government policy and macro-economic conditions may be uncertain. There may also be uncertainty in the total cost of an investment and the lead time between an investment decision and its implementation.

This paper is based on a participant model with a plant state which changes with time as a function of the participant's decision and the outcome of a stochastic process which represents the uncertainty phenomena. The range of decisions, the profits and the future state depend on the present state. The time dependent nature of uncertainty and the inter-temporal linking effects within the industry are thus represented. This work does not rely on the existence of a "long run" state at which investment has reach some optimal level. Such an assumption would not realistically represent the current dynamic and uncertain condition of many industries (particularly electricity).

In this model, the individual participant is presented with a pricing structure which determines how much they pay or are paid for consuming or producing industry products. The pricing structure can change with time in a non-predetermined fashion and thus future values are uncertain, as in spot pricing regimes. The participant must therefore be given a forecast of pricing future structures. The term *tariff* is used here to include the pricing structure and its forecast.

It is assumed that at each time point, the participant chooses decisions so as to maximize the sum of its immediate and expected future profits.

The main result presumes the existence of a centralized price setter which aims to induce behaviour from each participant such that the sum of present and expected future welfare is maximized. Further, total production must equal consumption for each industry product. Although losses and constraints due to transport (such as transmission and distribution losses and limits imposed by voltage restrictions in an electrical energy system) are ignored in this paper, the existing theory can easily be used to accommodate them.

The optimal tariff has two terms. The first is similar to the traditional short run marginal cost (SRMC) pricing where each unit of a product is traded at its SRMC [1]. However, some re-definition of SRMC is required to account for inter-temporal linking and uncertainty.

The second is a non-linear incentive term which exposes the participant to the effect its decisions have on the future profits of all participants, via its effect on the forecast of future pricing structures. It is different for each participant and tends to be insignificant for participants with small inter-temporal linkage phenomena.

The contribution of this work to pricing theory is the use of a general but detailed representation of uncertainty and inter-temporal linkages. Over the past two decades, significant advances have been made on the original work of Brown and Johnson [2] in dealing with uncertainty (see, for example, [3], [4], [5], [6], [7], [8] and [9]). However, in none of these works has the dynamic evolution of a participant's plant over time been adequately represented. Instead, consumers have been modelled as simple deterministic or random responders to price, sometimes with inter-dependence between consumption in intervals in a periodically repeated day. Thus the full range of supply and demand side responses to evolving uncertainty cannot be captured. The work of Caramanis, Bohn and Schweppe ([10], [11]) began a move in the direction of more complete modelling of time evolving behaviour. The current work, however, uses a more complete model.

The detailed representation of the inter-temporal linking *of decisions produces a new result which excludes capital* charges and demand charges. The incentive term has not been reported in the literature before. Specific examples have shown that it can be non-trivial for participants with significant inter-temporal effects and that it is essential for welfare maximization [12].

In section 2, the details of the individual participant model are given. Section 3 characterizes participant profit maximizing behaviour. The conditions for welfare maximization are developed in section 4 and the main result is stated and discussed in section 5 and proved in section 6. Section 7 concludes the paper with some limitations of the present work and some remarks about applying the results. A list of symbols used in this paper is included after section 7.

2. Model of the Participants

The model is based on discrete decision time points $t = 1,\ldots,T$. Define the set of decision time points as $\mathbf{T_d} = \{1,\ldots,T\}$. The time interval from time t to t+1 is referred to as the t^{th} time period and it is assumed that decisions affecting it are made at time t. Time $t = T+1$ is the end of the industry.

There are assumed to be J participants, labeled $j = 1,\ldots,J$. The set of all participants will be referred to as $\mathbf{J}$, defined by $\mathbf{J} = \{1,\ldots,J\}$. Each participant either consumes or produces one or more of P industry products, referred to as $p = 1,\ldots,P$. Define $\mathbf{P} = \{1,\ldots,P\}$.

At each time t, a participant j is characterized by the following entities:

- plant state vector
- random effects variable
- decision vector and a set of restrictions on it
- state dynamics
- industry products nett consumptions function
- benefits function
- pricing structure and forecast

In the following, each of these is defined.

Consider a particular participant $j \in \mathbf{J}$, at some decision time $t \in \mathbf{T_d}$.

The *Plant State Vector*, $x_j(t)$, stores information about the way in which the past history of the plant affects the present operation. It is assumed to be an M_j^x-vector with real components. For example, components of $x_j(t)$ might represent the level of storage of a particular substance, the stage reached in a start-up or shut-down sequence, the installed capacity of a plant item, the percentage completion of a plant installation programme or the level of funds available for investment.

The *Random Effects*, $\eta_j(t)$, summarizes the random phenomena which affect participant j during the t^{th} period. It does not include random variations in the pricing structure of industry products (these are dealt with elsewhere in the model). It is assumed that j knows $\eta_j(t)$ at decision time t but that future values $\eta_j(t+1),\ldots,\eta_j(T)$ are unknown. Instead, j has a probability distribution for these random variables which is a function of the observed value of $\eta_j(t)$. Further, the following assumption is made.

Assumption A1: The stochastic process $\{\eta_j(t)\}_{t=1}^{T}$ is Markov [13].

The random variable $\eta_j(t)$ does not necessarily have to be a vector of real numbers. Thus, $\eta_j(t)$ can represent any uncertainty, other than in the price of industry products. For example, it might represent a break-down or repair of a plant item, random fluctuations in the cost of some factor of production (other than an industry product), and uncertainty in the lead time of plant installation and costs. Note that, for $j \neq k$, the random variables $\eta_j(t)$ and $\eta_k(t)$ are ***not*** assumed to be independent, to allow for shared random effects (e.g. participants j and k both using the same factor of production).

The *Decision Vector,* $u_j(t)$, represents all the operational and investment decisions taken by participant j at decision time t. It is assumed that $u_j(t)$ is an M_j^u vector with real components. A component might, for example, represent the amount of consumption or production of a particular industry product during the period between times t and t+1, the operation of any internal storages and the amount of investment to be committed on a given type of plant. It is assumed that a decision $u_j(t)$ is taken after observing $x_j(t)$ and $\eta_j(t)$.

In this discrete time model, it is assumed that the condition of the participant's plant remains constant between decision points t and t+1 and that it is known at time t. The approximation introduced by this assumption can be minimized by making the period between decision time arbitrarily small.

Restrictions on the decision vector are imposed by, for example, plant capacity, storage sizes and the availability of investment options. They are represented by

$$h_j\left(x_j(t),u_j(t),\eta_j(t),t\right) \leqq 0 \qquad (1)$$

where $h_j\left(\cdot,t\right)$ is an M_j^h real-vector valued function, with components $h_{j,m}\left(\cdot,t\right)$, for $m = 1,\dots,M_j^h$. The symbol $\leqq$ implies component wise inequality so that equation (1) is equivalent to

$$h_{j,m}\left(x_j(t),u_j(t),\eta_j(t),t\right) \leq 0 \qquad (2)$$

for $m = 1,\ldots,M_j^h$. The dependence of $h_j(\bullet,t)$ on $x_j(t)$ models, for example, the dependence of the operating range of the plant on the cumulative effect of past decisions such as the amount of realized capital investment.

State dynamics model how the state vector changes with time as a function of random effects and decisions. The equation is

$$x_j(t+1) = f_j\big(x_j(t),u_j(t),\eta_j(t),t\big) \tag{3}$$

This equation captures the way in which operations and investment decisions at time t and the outcome of the random effects in the t^{th} period affect future conditions.

Nett Consumption Function. The decision vector $u_j(t)$ is assumed to uniquely specify the amount of each industry product consumed or produced by j in the t^{th} period. Thus define for $p \in \mathbf{P}$

$y_{j,p}\big(u_j(t),t\big)$ = nett consumption of product p in the t^{th} period

If $y_{j,p}\big(u_j(t),t\big)$ is positive, then participant j is a nett consumer of product p in the t^{th} period and if it is negative then j is a nett producer.

Define $y_j\big(u_j(t),t\big)$ as the P-vector with p^{th} component $y_{j,p}\big(u_j(t),t\big)$.

The Pricing Structure determines the cost charged to participant j for its decisions. In this general model of the participant, the charge can be a non-linear function of $u_j(t)$,

$c_j\big(u_j(t),t\big)$ = nett charge imposed for industry participation in t^{th} interval

Thus if $c_j\big(u_j(t),t\big)$ is positive, it represents a charge against participant j, while if it is negative, it represents income to j. This includes but is more general than a simple price-based charge on the components of $y_j\big(u_j(t),t\big)$.

Pricing Structure Forecasts. At decision time t, participant j knows $c_j\big(u_j(t),t\big)$ but the value of future structures, $\{c_j(\bullet,\tau) : \tau = t+1,\ldots,T\}$ are uncertain: only probability distributions are known. To model this situation, define

$$\phi(t) = \text{the pricing information state}$$

which is announced before participant j chooses $u_j(t)$. We suppose that j possesses a set of functions $\{\chi_j(\cdot,\tau) : \tau = 1,\ldots,T\}$ which interprets the forecast to give the current pricing structure, i.e.

$$\chi_j(\phi(t),t) = c_j(\cdot,t) \qquad (4)$$

Further, it is assumed that the process $\{\phi(t): t = 1,\ldots,T\}$ is Markov and that the transition probabilities are announced and known to participant j. Thus, once $\phi(t)$ has been announced, a participant can calculate the pricing structure for the t^{th} period and probability distributions on the structures for future periods.

The Benefits of participation for j in the t^{th} time period are the profits earned from any sales other than industry products minus the cost of purchase of any factors of production including capital items but excluding industry products. This includes all costs and incomes not covered by $c_j(\cdot,t)$. Thus, define

$b_j(x_j(t),u_j(t),\eta_j(t),t)$ = nett income earned in t^{th} period ex industry product purchases and/or sales

= income from sale of goods other than industry products (including scrapping of capital items)

minus

cost of purchase of goods other than industry products (including capital items)

Assumption A2: The functions $b_j(\cdot,\eta_j(t),t)$, $c_j(\cdot,t)$, $f_j(\cdot,\eta_j(t),t)$, $h_j(\cdot,\eta_j(t),t)$ and $y_j(\cdot,t)$ of $(x_j(t),u_j(t))$ are continuously differentiable.

3. Profit Maximizing Behaviour

At decision time t, participant j makes a profit which is its benefits less the nett charge for industry products. Thus the profit, $\pi_j(t)$ is defined as

$$\pi_j(t) = b_j\left(x_j(t), u_j(t), \eta_j(t), t\right) - c_j\left(u_j(t), t\right) \tag{5}$$

Participant j is assumed to observe $x_j(t), \eta_j(t)$ and $\phi(t)$ and then to choose $u_j(t)$ to maximize its immediate profits plus its expected future profits i.e. to maximize

$$NR_j(t) = \pi_j(t) + \mathbf{E}\left[\sum_{\tau=t+1}^{T} \pi_j(\tau) \;\middle|\; \eta_j(t), \phi(t)\right] \tag{6}$$

subject to equations (1) and (3) at time t and t+1,...,T.

The decision, $u_j(t)$, will affect the future plant states via equation (3) and will thus change the range of decisions open at future times. In equation (6), it is assumed that all future decisions will be chosen optimally as a function of the participant's plant state at that time. This is the "Principle of Optimality" of dynamic programming [14], which will be used here to develop conditions which describe the optimal behaviour of participant j.

In the following, the real valued functions $R_j\left[x_j(t), \eta_j(t), \phi(t), t\right]$ for t = T+1,T,...,1, are defined using induction. These will be equal to the optimised value of $NR_j(t)$.

For all admissible $x_j(T+1)$, $\eta_j(T+1)$ and $\phi(T+1)$, define

$$R_j\left[x_j(T+1), \eta_j(T+1), \phi(T+1), T+1\right] = 0 \tag{7}$$

Consider the problem of defining $R_j[\bullet, t]$ for some $t \le T$ and suppose that the function $R_j[\bullet, t+1]$ has already been defined. Further, suppose that participant j observes at decision time t that

$$x_j(t) = \bar{x}_j(t) \tag{8}$$

$$\eta_j(t) = \bar{\eta}_j(t) \tag{9}$$

$$\phi(t) = \bar{\phi}(t) \tag{10}$$

where $\bar{x}_j(t)$, $\bar{\eta}_j(t)$ and $\bar{\phi}(t)$ are, respectively, any admissible plant state vector, any realization (outcome) of the random effects process $\eta_j(t)$, and any pricing information state (i.e. a realization of $\phi(t)$). The over-bar is used to distinguish between the process and a realization of it. The following leads to a definition of $R_j[\bar{x}_j(t), \bar{\eta}_j(t), \bar{\phi}(t), t]$.

Define, for any decision vector, $u_j(t)$,

$$\bar{h}_j(u_j(t), t) = h_j(\bar{x}_j(t), u_j(t), \bar{\eta}_j(t), t) \tag{11}$$

Here and in the following, the over-bar on a function implies that the omitted arguments are evaluated at their over-bar values as in equation (8) to (10).

For any $u_j(t)$ satisfying the constraints on it imposed by equation (1) i.e.

$$\bar{h}_j(u_j(t), t) \leqq 0 \tag{12}$$

define

$$\bar{f}_j(u_j(t), t) = f_j(\bar{x}_j(t), u_j(t), \bar{\eta}_j(t), t) \tag{13}$$

$$\bar{b}_j(u_j(t), t) = b_j(\bar{x}_j(t), u_j(t), \bar{\eta}_j(t), t) \tag{14}$$

Further, for each realization of the pricing information state at decision time t+1, $\bar{\phi}(t+1)$, define

$$\begin{aligned} &\bar{r}_j(u_j(t), \bar{\phi}(t+1), t+1) \\ &\qquad = R_j[\bar{f}_j(u_j(t), t), \eta_j(t+1), \bar{\phi}(t+1), t+1] \end{aligned} \tag{15}$$

Remark

The random variable $\bar{r}_j(u_j(t), \bar{\phi}(t+1), t+1)$ can be interpreted as the maximum value of

the profit in the $t+1^{st}$ time period

plus

the expected profit from the $t+2^{nd}$ until the last time periods,

given a pricing information state at t+1 of $\bar{\phi}(t+1)$, and that $x_j(t+1) = \bar{f}_j(u_j(t), t)$ i.e. the plant state that results from decision vector $u_j(t)$.

The random variable $\bar{r}_j(u_j(t),\bar{\phi}(t+1),t+1)$ is adapted on $\eta_j(t+1)$ and the random variable $\bar{r}_j(u_j(t),\phi(t+1),t+1)$ is adapted on $\eta_j(t+1)$ and $\phi(t+1)$. ♦

Define $R_j[\bullet,t]$ by the following maximization of the profit in the t^{th} time interval plus the expected future profit:

$$R_j[\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t]$$

$$= \max\Big\{ \bar{b}_j(u_j(t),t) - c_j(u_j(t),t) + \mathbf{E}\big[\bar{r}_j(u_j(t),\phi(t+1),t+1) \,\big|\, \bar{\eta}_j(t),\bar{\phi}(t)\big] : \bar{h}_j(u_j(t),t) \leqq 0 \Big\} \quad (16)$$

The following assumption avoids ambiguities arising from the possibility of multiple local maxima in equation (16).

Assumption A3: For all $t \in \mathbf{T_d}$, and for all admissible $\bar{x}_j(t)$, $\bar{\eta}_j(t)$ and $\bar{\phi}(t)$, the functions $\bar{b}_j(\bullet,t)$, $c_j(\bullet,t)$ and $\bar{h}_j(\bullet,t)$ posses sufficient convexity such that the optimization in equation (16) has a single local maximum and no other stationary points.

Assumptions A2 and A3 imply that the first order Kuhn-Tucker conditions [15] for equation (16) will have a unique solution and that solution will be the maximizer in equation (16). These conditions are now used to develop an alternative characterization of $R_j[\bullet,t]$.

To this end first define

$$\bar{\rho}_j(u_j(t),t+1) = \mathbf{E}\big[\bar{r}_j(u_j(t),\phi(t+1),t+1) \,\big|\, \bar{\eta}_j(t),\bar{\phi}(t)\big] \quad (17)$$

Remark

This is the expected future profits (see equation (16)). ♦

The Lagrangian for equation (16) is

$$\bar{L}_j\left(u_j(t),\lambda_j(t),t\right) = \bar{b}_j\left(u_j(t),t\right) - c_j\left(u_j(t),t\right) + \bar{\rho}_j\left(u_j(t),t+1\right) + \lambda_j(t)\ \bar{h}_j\left(u_j(t),t\right) \tag{18}$$

The over-bar on $\bar{L}_j$ indicates a dependence on $\bar{x}_j(t)$, $\bar{\eta}_j(t)$ and $\bar{\phi}(t)$. The multiplier $\lambda_j(t)$ is an M_j^h-vector of real numbers, with components $\lambda_{j,m}(t)$ for $m = 1,\ldots,M_j^h$. The product $\lambda_j(t)\ \bar{h}_j\left(u_j(t),t\right)$ is thus interpreted as an inner product, i.e.

$$\lambda_j(t)\ \bar{h}_j\left(u_j(t),t\right) = \sum_{m=1}^{M_j^h} \lambda_{j,m}(t)\ \bar{h}_{j,m}\left(u_j(t),t\right) \tag{19}$$

The first order Kuhn-Tucker conditions are:

$$\frac{\partial \bar{b}_j\left(u_j(t),t\right)}{\partial u_j(t)} - \frac{\partial c_j\left(u_j(t),t\right)}{\partial u_j(t)} + \frac{\partial \bar{\rho}_j\left(u_j(t),t+1\right)}{\partial u_j(t)} + \lambda_j(t)\ \frac{\partial \bar{h}_j\left(u_j(t),t\right)}{\partial u_j(t)} = 0 \tag{20}$$

and for $m = 1,\ldots,M_j^h$

$$\lambda_{j,m}(t) = \begin{cases} =0 & \text{if } \bar{h}_{j,m}\left(u_j(t),t\right) < 0 \\ \leq 0 & \text{if } \bar{h}_{j,m}\left(u_j(t),t\right) = 0 \end{cases} \tag{21}$$

Remarks

1. Equation (20) is an M_j^u-vector equation which is equivalent to the M_j^u scalar equations,

$$\frac{\partial \bar{b}_j(u_j(t),t)}{\partial u_{j,n}(t)} - \frac{\partial c_j(u_j(t),t)}{\partial u_{j,n}(t)} + \frac{\partial \bar{\rho}_j(u_j(t),t+1)}{\partial u_{j,n}(t)} + \sum_{m=1}^{M_j^h} \lambda_{j,m}(t) \frac{\partial \bar{h}_{j,m}(u_j(t),t)}{\partial u_{j,n}(t)} = 0 \tag{22}$$

   for $n = 1,\dots,M_j^u$.

2. Equation (21) is the complementary slackness condition. ♦

Under the above assumptions, equations (20) and (21) will have a unique solution for $u_j(t)$ and $\lambda_j(t)$, which will be referred to as

$$\tilde{u}_j(t) = \tilde{u}_j(\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t) \tag{23}$$

$$\tilde{\lambda}_j(t) = \tilde{\lambda}_j(\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t) \tag{24}$$

In summary, the inductive definition of the functions $R_j[\bullet,t]$ is as follows. Given the function $R_j[\bullet,t+1]$, for each admissible realization $\bar{x}_j(t)$, $\bar{\eta}_j(t)$, $\bar{\phi}(t)$ define $\tilde{u}_j(t)$ and $\tilde{\lambda}_j(t)$ as the unique solution of equations (20) and (21). This value of $\tilde{u}_j(t)$ will be j's profit maximizing decision at time t for these realizations. Then define

$$R_j[\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t] = \bar{b}_j(\tilde{u}_j(t),t) - c_j(\tilde{u}_j(t),t) + \bar{\rho}_j(\tilde{u}_j(t),t+1) \tag{25}$$

4. Welfare Maximizing Behaviour

The objective is to induce all participants to behave in a way in which global welfare is maximized. In this section, conditions which characterize socially optimal behaviour are developed.

At some time $t \in \mathbf{T_d}$, the social benefit of each participant making a decision $u_j(t)$ is

$$\sigma(t) = \sum_{j \in \mathbf{J}} b_j\big(x_j(t), u_j(t), \eta_j(t), t\big) \tag{26}$$

being the sum of the benefits to each participant. The incomes and outlays on industry products are excluded from the definition of $\sigma(t)$ since these are transfer payments.

Thus the socially optimal decisions are those which maximize the social welfare defined as the social benefit plus expected future social benefit i.e.

$$S(t) = \sigma(t) + \mathbf{E}\Big[\sum_{\tau=t+1}^{T} \sigma(\tau) \;\Big|\; \eta_j(t),\ j \in \mathbf{J} \Big] \tag{27}$$

subject to the constraints on each participant $j \in \mathbf{J}$ at time $\tau = t,\ldots,T$

$$h_j\big(x_j(\tau), u_j(\tau), \eta_j(\tau), \tau\big) \leqq 0 \tag{28}$$

and to the industry products balance equation for $\tau = t,\ldots,T$

$$\sum_{j \in \mathbf{J}} y_j\big(u_j(\tau), \tau\big) = 0 \tag{29}$$

Remarks.

1. Equation (29) is an P-vector equation, with components for $p \in \mathbf{P}$,

$$\sum_{j \in \mathbf{J}} y_{j,p}\big(u_j(\tau), \tau\big) = 0 \tag{30}$$

   Thus (29) implies that for each product at each time, the amount produced equals the amount consumed.

2. Equation (29) assumes there are no losses in transporting product p between participants and no

constraints imposed by the transportation process. These can easily be accounted for.

♦

In the following, conditions on $\{u_j(t) : j \in \mathbf{J}\}$ for social optimality are derived using a dynamic programming approach, similar to that used to develop the characterization of the individual profit maximizing participant in section 2 above.

First the following notational convention simplifies the presentation. For any quantity $v_j(t)$ defined for each $j \in \mathbf{J}$ such as $x_j(t)$, $u_j(t)$, $\eta_j(t)$, $\bar{x}_j(t)$ and $\bar{\eta}_j(t)$, define

$$v(t) = \left(v_1(t), \ldots, v_j(t), \ldots, v_J(t)\right) \tag{31}$$

The vector $v(t)$ will be referred to as the social version of $v_j(t)$.

For any admissible $\bar{x}(t)$, $\bar{\eta}(t)$, define for each admissible $u(t)$

$$\bar{f}\left(u(t),t\right) = \left(\bar{f}_1\left(u_1(t),t\right), \ldots, \bar{f}_j\left(u_j(t),t\right), \ldots, \bar{f}_J\left(u_J(t),t\right)\right) \tag{32}$$

Thus

$$x(t+1) = \bar{f}\left(u(t),t\right) \Leftrightarrow x_j(t+1) = \bar{f}_j\left(u_j(t),t\right) \quad \forall \; j \in \mathbf{J} \tag{33}$$

The real valued functions $W[x(t),\eta(t),t]$ are now defined inductively for $t = T+1, T, \ldots, 1$. These will be equal to $S(t)$.

For all admissible $x(T+1)$ and $\eta(T+1)$ define

$$W[x(T+1),\eta(T+1),T+1] = 0 \tag{34}$$

Consider the problem of defining $W[\bullet,t]$ for some $t \in \mathbf{T_d}$ and suppose that the function $W[\bullet,t+1]$ has already been defined. Further, suppose that

$$x(t) = \bar{x}(t) \tag{35}$$

$$\eta(t) = \bar{\eta}(t) \tag{36}$$

The following leads to a definition of $W[\bar{x}(t),\bar{\eta}(t),t]$.

For any u(t) such that for all $j \in \boldsymbol{J}$,

$$\bar{h}_j\big(u_j(t),t\big) \leqq 0 \qquad (37)$$

define

$$\bar{\omega}\big(u(t),t+1\big) = \mathbf{E}\Big[W\big[\bar{f}\big(u(t),t\big),\eta(t+1),t+1\big] \;\Big|\; \bar{\eta}(t)\Big] \qquad (38)$$

Remark

> This can be interpreted as the expected value of the future social benefits (summed over t+1,...,T) given that the social random effects process is $\bar{\eta}(t)$ (i.e. $\eta_j(t) = \bar{\eta}_j(t)$) and that the social state at t+1 is given by $x(t+1) = \bar{f}\big(u(t),t\big)$. ♦

Then define

$$W\big[\bar{x}(t),\bar{\eta}(t),t\big] = \max\Big\{ \sum_{j\in\boldsymbol{J}} \bar{b}_j\big(u_j(t),t\big) + \bar{\omega}\big(u(t),t+1\big) :$$

$$\bar{h}_j\big(u_j(t),t\big) \leqq 0 \quad j\in\boldsymbol{J}$$

$$\sum_{j\in\boldsymbol{J}} y_j\big(u_j(t),t\big) = 0 \Big\} \qquad (39)$$

As in the participant's profit maximization problem, assumptions A2 and A3 imply that the first order Kuhn-Tucker conditions have a unique solution which is the maximizer for equation (39).

The Lagrangian for this problem is

$$\bar{L}^G\big(u(t),\lambda(t),\theta(t),t\big)$$

$$= \sum_{j\in\boldsymbol{J}} \bar{b}_j\big(u_j(t),t\big) + \bar{\omega}\big(u(t),t+1\big)$$

$$+\sum_{j\in\boldsymbol{J}} \lambda_j(t)\, \bar{h}_j\big(u_j(t),t\big) + \theta(t) \sum_{j\in\boldsymbol{J}} y_j\big(u_j(t),t\big) \qquad (40)$$

The multipliers $\lambda_j(t)$ are M_j^h-vectors of real numbers, with components $\lambda_{j,m}(t)$ for $m = 1,\ldots,M_j^h$. The multiplier $\theta(t)$ is a P-vector and $\theta(t) \sum_{j\in \mathbf{J}} y_j\left(u_j(t),t\right)$ is the inner-product

$$\sum_{p\in \mathbf{P}} \left\{\theta_p(t) \sum_{j\in \mathbf{J}} y_{j,p}\left(u_j(t),t\right)\right\}$$

The first order Kuhn-Tucker conditions are:

For all $j \in \mathbf{J}$

$$\frac{\partial \bar{b}_j\left(u_j(t),t\right)}{\partial u_j(t)} + \frac{\partial \bar{\omega}_j\left(u(t),t+1\right)}{\partial u_j(t)}$$

$$+ \lambda_j(t) \frac{\partial \bar{h}_j\left(u_j(t),t\right)}{\partial u_j(t)} + \theta(t) \frac{\partial y_j\left(u_j(t),t\right)}{\partial u_j(t)} = 0 \qquad (41)$$

and

$$\sum_{j\in \mathbf{J}} y_j\left(u_j(t),t\right) = 0 \qquad (42)$$

and the complementary slackness condition; i.e. for each $j \in \mathbf{J}$ and for all $m = 1,\ldots,M_j^h$

$$\lambda_{j,m}(t) = \begin{cases} =0 & \text{if } \bar{h}_{j,m}\left(u_j(t),t\right) < 0 \\ \leq 0 & \text{if } \bar{h}_{j,m}\left(u_j(t),t\right) = 0 \end{cases} \qquad (43)$$

Under the above assumptions, equations (41), (42) and (43) will have a unique solution, which will be referred to as

$$\hat{u}_j(t) = \hat{u}_j\left(\bar{x}(t),\bar{\eta}(t),t\right) \qquad (44)$$

$$\hat{\lambda}_j(t) = \hat{\lambda}_j\left(\bar{x}(t),\bar{\eta}(t),t\right) \qquad (45)$$

$$\hat{\theta}(t) = \hat{\theta}\left(\bar{x}(t),\bar{\eta}(t),t\right) \qquad (46)$$

In summary, the inductive definition of the functions $W[\bullet,t]$ is as follows. Given the function $W[\bullet,t+1]$, for each admissible $\bar{x}(t)$ and $\bar{\eta}(t)$, define $\hat{u}(t)$, $\hat{\lambda}(t)$ and $\hat{\theta}(t)$ as the unique solution of equations (42), (42) and (43). This value of $\hat{u}_j(t)$ will be the optimal decision at time t. Then define

$$W[\bar{x}(t),\bar{\eta}(t),t] = \sum_{j\in \mathbf{J}} \bar{b}_j\left(\hat{u}_j(t),t\right) + \bar{\omega}\left(\hat{u}(t),t+1\right) \tag{47}$$

Remark

Note that $\hat{u}_j(t)$ depends on $\left(\bar{x}_k(t),\bar{\eta}_k(t)\right)$ for all $k \in \mathbf{J}$. However, $\tilde{u}_j(t)$ depends on $\left(\bar{x}_j(t),\bar{\eta}_j(t)\right)$ and $\bar{\phi}(t)$ but not $\left(\bar{x}_k(t),\bar{\eta}_k(t)\right)$ for $k \neq j$. ♦

5. Optimal Pricing: The Main Result

The aim is to set the pricing structure and forecast so that the response of the profit maximizing participant is the same as the socially optimal response. That is, to design a rule $\psi(\cdot,t)$ such that if

$$\phi(t) = \psi(\bar{x}(t),\bar{\eta}(t),t) \tag{48}$$

then for all $j \in \boldsymbol{J}$

$$\tilde{u}_j(\bar{x}_j(t),\bar{\eta}_j(t),\phi(t),t) = \hat{u}_j(\bar{x}(t),\bar{\eta}(t),t) \tag{49}$$

In the following, conditions on the function $\psi(\cdot,t)$ such that equation (49) holds are stated inductively and then discussed. The proof is delayed until the next section.

Consider some $t \in \mathbf{T_d}$. Suppose that $\psi(\bar{x}(t+1),\bar{\eta}(t+1),t+1)$ has been defined for all admissible $\bar{x}(t+1)$ and $\bar{\eta}(t+1)$ and consider some admissible $\bar{x}(t)$ and $\bar{\eta}(t)$. Let $\hat{\theta}(t)$ and $\hat{u}(t)$ be defined as the solution of the global welfare maximizing problem above. That is, for all $j \in \boldsymbol{J}$

$$\hat{u}_j(t) = \hat{u}_j(\bar{x}(t),\bar{\eta}(t),t) \tag{50}$$

$$\hat{\theta}(t) = \hat{\theta}(\bar{x}(t),\bar{\eta}(t),t) \tag{51}$$

For any $u(t)$ such that for all $j \in \boldsymbol{J}$

$$\bar{h}_j(u_j(t),t) \leqq 0 \tag{52}$$

define

$$\bar{\psi}(u(t),t+1) = \psi[\bar{f}(u(t),t),\eta(t+1),t+1] \tag{53}$$

Remark

> This is the pricing information announced under the ψ-rule at decision time t+1 if each participant j makes a decision $u_j(t)$ at time t so that $x(t+1) = \bar{f}(u(t),t)$. It is a random variable adapted on $\eta(t+1)$. ♦

Further define

$$\hat{U}^j\big(u_j(t),t\big) = \big(\hat{u}_1(t),\ldots,\hat{u}_{j-1}(t),u_j(t),\hat{u}_{j+1}(t),\ldots,\hat{u}_J(t)\big) \tag{54}$$

and

$$\hat{\psi}_j\left[u_j(t),t+1\right] = \bar{\psi}\left[\hat{U}^j\big(u_j(t),t\big),t+1\right] \tag{55}$$

Remark

This is the pricing information for decision time t+1 if all participants k, other than j make decision $\hat{u}_k(t)$ at decision time t and participant j makes decision $u_j(t)$. It is thus the pricing information based on $x(t+1) = \bar{f}\left[\hat{U}^j\big(u_j(t),t\big),t\right]$. It is a random variable adapted on $\eta(t+1)$. ♦

Finally define for $j,k \in \mathbf{J}$

$$\hat{r}_k^j\big(u_j(t),t+1\big) = \bar{r}_k\big(\hat{u}_k(t),\hat{\psi}_j\left[u_j(t),t+1\right],t+1\big) \tag{56}$$

Remark

The real random variable $\hat{r}_k^j\big(u_j(t),t+1\big)$ is the profit in the t+1st interval plus the expected profit in subsequent intervals given that:

1. The state of participant k is based on socially optimal decisions for times t,...,T, so that $x_k(t+1) = \bar{f}_k\big(\hat{u}_k(t),t\big)$.
2. The price information at time t+1 is based on $x(t+1) = \bar{f}\left[\hat{U}^j\big(u_j(t),t\big),t\right]$ i.e. all participants other than j making socially optimal decisions and j making decision $u_j(t)$.

Thus, for $k \neq j$, $\hat{r}_k^j\big(u_j(t),t+1\big)$ measures the effect that participant j's decision at time t has on participants k's profits on t+1,...,T+1, through its effect on the forecasts, under the assumption that all other decisions are socially optimal. For k = j, $\hat{r}_j^j\big(u_j(t),t+1\big)$ measures the effect that $u_j(t)$ has on j's future profits, through its effect on forecasts, but not through its direct effects on j's future states. ♦

Then in order to be optimal, the rule $\psi(\bullet,t)$ must produce price information

$$\bar{\phi}(t) = \psi(\bar{x}(t),\bar{\eta}(t),t) \tag{57}$$

such that the following two conditions hold.

CONDITION I: For each participant $j \in \mathbf{J}$, the tariff associated with $\bar{\phi}(t)$ at decision time t

$$\chi_j(\bar{\phi}(t),t) = c_j(\bullet,t) \tag{58}$$

is given by

$$\begin{aligned} c_j(u_j(t),t) &= -\hat{\theta}(t)\; y_j(u_j(t),t) \\ &+ \sum_{k\in \mathbf{J}} \mathbf{E}\big[\hat{r}_k^j(u_j(t),t+1)\;\big|\;\bar{\eta}_k(t),\bar{\phi}(t)\big] \\ &- \sum_{k\in \mathbf{J}} \mathbf{E}\big[\hat{r}_k^j(\hat{u}_j(t),t+1)\;\big|\;\bar{\eta}_k(t),\bar{\phi}(t)\big] \end{aligned} \tag{59}$$

CONDITION II. For any u(t) such that for all $j \in \mathbf{J}$

$$\bar{h}_j(u_j(t),t) \leqq 0 \tag{60}$$

define

$$\bar{\phi}(t+1) = \bar{\psi}(u(t),t+1) \tag{61}$$

For any Borel measurable [16] functions α and β

$$\mathbf{E}\big[\alpha(\eta(t+1))\;\big|\;\bar{\eta}(t),\bar{\phi}(t)\big] = \mathbf{E}\big[\alpha(\eta(t+1))\;\big|\;\bar{\eta}(t)\big] \tag{62}$$

and

$$\mathbf{E}\big[\beta(\eta_k(t+1),\bar{\phi}(t+1))\;\big|\;\bar{\eta}(t),\bar{\phi}(t)\big]$$

$$= \mathbf{E}\left[\beta\left(\eta_k(t+1),\bar{\phi}(t+1)\right) \;\middle|\; \bar{\eta}_k(t),\bar{\phi}(t)\right] \tag{63}$$

Remarks

1. Condition I specifies the optimal tariff while condition II is the requirements on the forecasting content of the price information.

2. The first term in the optimal tariff in equation (59), the term

$$-\hat{\theta}(t)\; y_j\left(u_j(t),t\right) = -\sum_{p\in\mathbf{P}} \hat{\theta}_p(t)\; y_{j,p}\left(u_j(t),t\right)$$

is similar to traditional short run marginal cost (SRMC) pricing [1]: each unit of product p consumed by a participant attracts a charge of $-\hat{\theta}_p(t)$ and each unit sent out earns an income of $-\hat{\theta}_p(t)$.

The multiplier $\hat{\theta}(t)$ can be interpreted as follows. Suppose that for some $p \in \mathbf{P}$ the entire industry is required to supply an additional ε_p units of product p during the t^{th} time interval, by reduced consumption or increased production by some participant(s). Further suppose that this is to be done in a socially optimal fashion; that is, so as to solve

$$W^*\left[\bar{x}(t),\bar{\eta}(t),\varepsilon_p,t\right]$$

$$= \max\left\{ \sum_{j\in\mathbf{J}} \bar{b}_j\left(u_j(t),t\right) + \bar{\omega}\left(u(t),t+1\right) : \right.$$

$$\bar{h}_j\left(u_j(t),t\right) \leqq 0 \quad j\in\mathbf{J},$$

$$\sum_{j\in\mathbf{J}} y_{j,q}\left(u_j(t),t\right) = 0 \quad q\in\mathbf{P} \quad q\neq p$$

$$\left.\sum_{j\in\mathbf{J}} y_{j,p}\left(u_j(t),t\right) = -\varepsilon_p \right\} \tag{64}$$

Then

$$- \hat{\theta}_p(t) = \frac{\partial}{\partial \varepsilon_p} W^* [\bar{x}(t), \bar{\eta}(t), \varepsilon_p, t] \Bigg|_{\varepsilon_p = 0} \quad (65)$$

which is the social cost of providing an incremental unit of product p.

Note that this marginal cost is short run because the incremental product is to be supplied in the t^{th} time interval only, even though some of the social costs might be incurred in later intervals (for example, if there are pumped storage generators.)

3. A decision by participant j at time t affects the future profits of all participants via two mechanisms. First, it affects its own profits by its effect on its own future states. Secondly, a decision taken by j at time t affects its state at time t+1 and hence the social state at t+1 and thus the price information at t+1. The second term in (59),

$$\sum_{k \in J} \mathbf{E}\left[\hat{r}_k^j \left(u_j(t), t+1\right) \middle| \bar{\eta}_k(t), \bar{\phi}(t)\right]$$

exposes participant j to the effects of its actions at time t on the expected future profits of all participants via the second mechanism.

For participants with no inter-temporal links (i.e. no storage and no investment options), this term will be zero and for those whose linkage terms represent a small fraction of the total in the system, it will be negligible.

Note that the tariff at the socially optimal consumption level equals the SRMC charge, i.e.

$$c_j\left(\hat{u}_j(t), t\right) = -\hat{\theta}(t)\; y_j\left(\hat{u}_j(t), t\right) \quad (66)$$

because the second and third terms cancel out. This does not imply that these terms are unnecessary in the optimal tariff. Without them, a profit maximizing participant might make decisions which result in a forecast which severely reduces future social welfare. Examples of this have been developed [12].

4. The theory presented in this paper can be modified to incorporate a fixed charge into the tariff without reducing global welfare. However, this additional charge must be independent of the participants' decisions and their state. The latter is important in order that a participant not attempt to alter its

future state to reduce the charge, thereby deviating from welfare maximizing behaviour.

5. Equation (62) implies that given the social random effects vector at time t, the price information at time t contains no additional information about the random effects vectors at time t+1.

6. Equation (63) states that, given the random effects vector at time t for a particular participant, the pricing information at t summarizes all the stochastic information that is relevant to that participant and that is contained in the other participants random effects vectors. ♦

Main Result: Under assumptions A1, A2 and A3, if the rule $\psi(\cdot,t)$ obeys conditions I and II, then for any admissible $\bar{x}_j(t)$ and $\bar{\eta}_j(t)$ if

$$\bar{\phi}(t) = \psi\left(\bar{x}(t),\bar{\eta}(t),t\right) \tag{67}$$

then for all $j \in \mathbf{J}$

$$\tilde{u}_j\left(\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t\right) = \hat{u}_j\left(\bar{x}(t),\bar{\eta}(t),t\right) \tag{68}$$

♦

6. Proof of Main Result

In the following, for notational convenience, ∇_j is written for the operator $\frac{\partial}{\partial u_j(t)}$.

The proof is based on establishing by induction that for all $t \in \mathbf{T_d}$, and for all admissible $\bar{x}(t)$ and $\bar{\eta}(t)$, if

$$\bar{\phi}(t) = \psi\left(\bar{x}(t), \bar{\eta}(t), t\right) \tag{69}$$

then

$$W[\bar{x}(t), \bar{\eta}(t), t] = \sum_{j \in \mathbf{J}} R_j[\bar{x}_j(t), \bar{\eta}_j(t), \bar{\phi}(t), t] \tag{70}$$

Equation (70) holds for t = T+1 since both sides are zero (see equations (7) and (34)). For some $t \in \mathbf{T_d}$, suppose as an induction hypothesis that for all admissible $\bar{x}(t+1)$ and $\bar{\eta}(t+1)$ if

$$\bar{\phi}(t+1) = \psi\left(\bar{x}(t+1), \bar{\eta}(t+1), t+1\right) \tag{71}$$

then

$$W[\bar{x}(t+1), \bar{\eta}(t+1), t+1]$$

$$= \sum_{j \in \mathbf{J}} R_j[\bar{x}_j(t+1), \bar{\eta}_j(t+1), \bar{\phi}(t+1), t+1] \tag{72}$$

The aim is to establish that equation (70) holds so that by induction it will hold for all t = T+1, T, ..., 1.

Consider any admissible $\bar{x}(t)$ and $\bar{\eta}(t)$ and any $u(t)$ such that $\forall\ j \in \mathbf{J}$

$$\bar{h}_j\left(u_j(t), t\right) \leqq 0 \tag{73}$$

Define

$$\bar{\phi}(t) = \psi\left(\bar{x}(t), \bar{\eta}(t), t\right) \tag{74}$$

In the induction hypothesis, (i.e. equation (72)), let

$$\bar{x}(t+1) = \bar{f}(u(t),t) \tag{75}$$

$$\bar{\eta}(t+1) = \eta(t+1) \tag{76}$$

and thus (equation (71))

$$\begin{aligned}\bar{\phi}(t+1) &= \psi[\bar{f}(u(t),t),\eta(t+1),t+1] \\ &= \bar{\psi}(u(t),t+1)\end{aligned} \tag{77}$$

Taking the expectation of both sides of equation (72), conditioned on $\bar{\eta}(t)$ and $\bar{\phi}(t)$ and using equations (62) and (63) yields

$$\bar{\omega}(u(t),t+1) = \sum_{k \in \mathbf{J}} \mathbf{E}\{\bar{r}_k[u_k(t),\bar{\psi}(u(t),t+1),t+1] \mid \bar{\eta}_k(t),\bar{\phi}(t)\} \tag{78}$$

Consider the solution to the welfare maximization problem at time t in equation (39) i.e. for all $j \in \mathbf{J}$

$$\hat{u}_j(t) = \hat{u}_j(\bar{x}(t),\bar{\eta}(t),t) \tag{79}$$

$$\hat{\lambda}_j(t) = \hat{\lambda}_j(\bar{x}(t),\bar{\eta}(t),t) \tag{80}$$

$$\hat{\theta}(t) = \hat{\theta}(\bar{x}(t),\bar{\eta}(t),t) \tag{81}$$

Define

$$\hat{\phi}(t+1) = \bar{\psi}(\hat{u}(t),t+1) \tag{82}$$

The variables $\hat{u}_j(t)$, $\hat{\lambda}_j(t)$ and $\hat{\theta}(t)$ will satisfy equations (41), (42) and (43). Thus for each $j \in \mathbf{J}$, $\hat{u}_j(t)$ and $\hat{\lambda}_j(t)$ satisfy

$$\begin{aligned}&\nabla_j\bar{b}_j(u_j(t),t) + \nabla_j\bar{\omega}_j(\hat{U}^j[u_j(t),t],t+1) \\ &+ \lambda_j(t)\ \nabla_j\bar{h}_j(u_j(t),t) + \hat{\theta}(t)\ \nabla_j y_j(u_j(t),t) = 0\end{aligned} \tag{83}$$

This follows from equation (41) with $\hat{u}_k(t)$ substituted for $u_k(t)$ for all $k \neq j$ and $\hat{\theta}(t)$ for $\theta(t)$.

Differentiating equation (78) with respect to $u_j(t)$ yields

$$\nabla_j \bar{\omega}\left(\hat{U}^j\left[u_j(t),t\right],t+1\right)$$

$$= \nabla_j \mathbf{E}\left\{\bar{r}_j\left[u_j(t),\bar{\phi}(t+1),t+1\right] \,\middle|\, \bar{\eta}_j(t),\bar{\phi}(t)\right\}$$

$$+ \nabla_j \sum_{k \in J} \mathbf{E}\left\{\bar{r}_k\left(v_k(t),\hat{\psi}_j\left[u_j(t),t+1\right],t+1\right) \,\middle|\, \bar{\eta}_k(t),\bar{\phi}(t)\right\} \quad (84)$$

where

$$\bar{\phi}(t+1) = \hat{\psi}_j\left[u_j(t),t+1\right] \quad (85)$$

and

$$v(t) = \hat{U}^j\left(u_j(t),t\right) \quad (86)$$

The variable $u_j(t)$ enters on the right hand side of equation (78) in two places: the first argument of the j^{th} summand and in $\bar{\psi}\left(u(t),t+1\right)$ which is in all the summands. The two terms in equation (78) are the derivatives with respect to each of these entries in turn, with the other being held constant. The equation follows from the chain rule of differentiation.

Equation (84) can be used to substitute for $\nabla_j \bar{\omega}\left(\hat{U}^j\left[u_j(t),t\right],t+1\right)$ in equation (83); $\hat{u}_j(t)$ and $\hat{\lambda}_j(t)$ will satisfy the resulting equation. This will remain so if $v(t)$ is replaced by $\hat{u}(t)$ (i.e. $v_j(t)$ replaced by $\hat{u}_j(t)$) in the second term in (84) and $\hat{\psi}_j\left[\hat{u}_j(t),t+1\right]$ can replace $\hat{\psi}_j\left[u_j(t),t+1\right]$ (i.e. $\hat{\phi}(t+1)$ replaces $\bar{\phi}(t+1)$) in the first term.

It thus follows that $\hat{u}_j(t)$ and $\hat{\lambda}_j(t)$ solve

$$\nabla_j \bar{b}_j(u_j(t),t) + \nabla_j \mathbf{E}\{\bar{r}_j[u_j(t),\hat{\phi}(t+1),t+1] \mid \bar{\eta}_j(t),\bar{\phi}(t)\}$$

$$+ \nabla_j \sum_{k\in \mathbf{J}} \mathbf{E}\{\hat{r}_k^j(u_j(t),t+1) \mid \bar{\eta}_k(t),\bar{\phi}(t)\}$$

$$+ \lambda_j(t)\ \nabla_j \bar{h}_j(u_j(t),t) + \hat{\theta}(t)\ \nabla_j y_j(u_j(t),t) = 0 \qquad (87)$$

Now defining

$$c_j(\cdot,t) = \chi_j(\bar{\phi}(t),t) \qquad (88)$$

and using equation (59), equation (87) can be re-written as

$$\nabla_j \bar{b}_j(u_j(t),t) + \nabla_j \mathbf{E}\{\bar{r}_j[u_j(t),\hat{\phi}(t+1),t+1] \mid \bar{\eta}_j(t),\bar{\phi}(t)\}$$

$$- \nabla_j c_j(u_j(t),t) + \lambda_j(t)\ \nabla_j \bar{h}_j(u_j(t),t) = 0 \qquad (89)$$

which is exactly equation (20). Now $\hat{u}_j(t)$ and $\hat{\lambda}_j(t)$ satisfy the complementary slackness condition in equation (43) which is identical to the individual consumer's complementary slackness condition in equation (21). Thus $\hat{u}_j(t)$ and $\hat{\lambda}_j(t)$ solve the Kuhn-Tucker conditions for the individual participant's profit maximization problem and by Assumption A3

$$\tilde{u}_j(\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t) = \hat{u}_j(\bar{x}(t),\bar{\eta}(t),t) \qquad (90)$$

$$\tilde{\lambda}_j(\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t) = \hat{\lambda}_j(\bar{x}(t),\bar{\eta}(t),t) \qquad (91)$$

This holds for all $j \in \mathbf{J}$. It thus follows that

$$\sum_{j\in \mathbf{J}} y_j(\tilde{u}_j(t),t) = \sum_{j\in \mathbf{J}} y_j(\hat{u}_j(t),t)$$

$$= 0 \qquad (92)$$

which follows from the balance equation (42) in the definition of $\hat{u}(t)$.

Also by equations (25) and (66)

$$R_j[\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t]$$

$$= \bar{b}_j(\hat{u}_j(t),t) - c_j(\hat{u}_j(t),t) + \bar{\rho}_j(\hat{u}_j(t),t+1)$$

$$= \bar{b}_j(\hat{u}_j(t),t) - \hat{\theta}(t)\; y_j(\hat{u}_j(t),t) + \bar{\rho}_j(\hat{u}_j(t),t+1) \qquad (93)$$

Substituting $\hat{u}(t)$ for $u(t)$ in equation (78) implies that

$$\bar{\omega}(\hat{u}(t),t+1) = \sum_{j\in J} \bar{\rho}_j(\hat{u}_j(t),t+1) \qquad (94)$$

Summing equation (93) and using equations (92) and (94) implies

$$\sum_{j\in J} R_j[\bar{x}_j(t),\bar{\eta}_j(t),\bar{\phi}(t),t]$$

$$= \sum_{j\in J} \bar{b}_j(\hat{u}_j(t),t) - \hat{\theta}(t) \sum_{j\in J} y_j(\hat{u}_j(t),t)$$

$$+ \sum_{j\in J} \bar{\rho}_j(\hat{u}_j(t),t+1)$$

$$= \sum_{j\in J} \bar{b}_j(\hat{u}_j(t),t) + \bar{\omega}(\hat{u}(t),t+1)$$

$$= W[\bar{x}(t),\bar{\eta}(t),t] \qquad (95)$$

where equation (47) was used to establish the latter equality.

Since equation (95) holds for all admissible $\bar{x}(t)$ and $\bar{\eta}(t)$, the induction hypothesis has been established. It thus follows that equation (90) holds for all admissible $\bar{x}(t)$ and $\bar{\eta}(t)$.

QED

7. Conclusions

The theory of welfare optimal prices developed in this paper includes the time related phenomena of uncertainty and inter-temporal linking. This is achieved by explicitly modeling the time variable. The resulting optimal tariff includes two terms: SRMC pricing and an incentive term to expose participants to the effects of their decisions on the future profits of other participants, via the forecast. The second term is new and results from the representation of the time domain. This paper specifically rejects the concept of a "long run" in which investment has reached some optimal state as being unrealistic for an industry with uncertainty.

Because the model used in this paper is very general, the results covers a large number of situations. However, some of the limitations of this work are now discussed.

First, the model is based on decisions being represented by a vector chosen from a continuous range. Thus indivisibility of investment and production or consumption is not represented. In the context of the electricity industry, this is particularly significant for the investment process, where a choice is often being made from a restricted group of different technologies (e.g. gas turbines, coal fired boilers, hydro.).

Each participant is assumed to be risk neutral in that the expected value of future profit appears in the objective function. A more realistic model of participant behaviour would allow for risk averse participants who place greater stress on avoiding low levels of future profit. Work in this direction is currently under way.

The amount of information required by the centralized price setter is unrealistic. It would be impossible to collect the states, current outcomes and the probability distributions of future values of each participant's random effects. Further work is required to assess the minimum information set and the welfare costs of having less.

The incentive term in the tariff would be administratively difficult. It would be necessary to convert it into a non-linear charging of consumption or production and a tax or bonus on investments.

Further work on these and other implementational difficulties is under way and preliminary results are reported in [17].

Despite these limitations, the result is a useful indication of the appropriate content of a public utility or regulated tariff. Demand charges and long run marginal cost pricing cause welfare losses. Despite difficulties in calculating SRMC, good approximations can be found and, for small consumers or private generators and co-generators, they will induce socially optimal behaviour. For larger participants whose behaviour can influence future system state and hence future values of SRMC, steps need to be taken to expose them to any adverse effects that their immediate decisions, particularly investment plans, might have on the future of other participants.

Notation

1. Conventions

An over-bar on a quantity (e.g. $\bar{x}_j(t)$, $\bar{\eta}_j(t)$) refers to a particular realization of that quantity. Omission of a subscript indicates that the entire vector is being referred to - see equations (31) and (32). The inequality $\leqq$ implies component wise inequality, i.e. for a K-vector v

$$v \leqq 0 \quad \Leftrightarrow \quad v_k \leq 0 \quad \forall \; k = 1,\ldots,K$$

2. List of Major Symbols

α, β	=	any Borel measurable functions
$b_j(\cdot,t)$	=	nett benefits earned in t^{th} period excluding income/outlays from industry products
$\bar{b}_j(\cdot,t)$	-	see equation (14)
$c_j(\cdot,t)$	=	nett charge imposed on participant j for industry participation in t^{th} interval
$\chi_j(\cdot,t)$	=	tariff information state interpretation function - see equation (4)
$\mathbf{E}[\ \mid\ ]$	-	the conditional expectation operator
$f_j(\cdot,t)$	=	function used to express state dynamics - see equation (3)
$\bar{f}_j(\cdot,t)$	-	see equation (13)
$\bar{f}(u(t),t)$		vector of $\bar{f}_j(u_j(t),t)$, $j \in \mathbf{J}$ - see equation (32)
$\phi(t)$	=	tariff information state
$\hat{\phi}(t+1)$	-	see equation (82) and (79)
$h_{j,m}(\cdot,t)$		function used to express restrictions on decision vector- see equations (1) and (2)
$h_j(\cdot,t)$	=	vector of $h_{j,m}(\cdot,t)$, $m = 1,\ldots,M_h^j$
$\bar{h}_j(\cdot,t)$	-	see equation (11)
$\eta_j(t)$	=	random effects for participant j in t^{th} time interval
J	=	number of participants
$\mathbf{J}$	=	$\{1,\ldots,J\}$, the set of participants
$\bar{L}^G(\cdot,t)$	-	see equation (40)
$\bar{L}_j(\cdot,t)$	-	see equation (18)
$\lambda_{j,m}(t)$	=	Kuhn-Tucker multiplier associated with the m^{th} constraint on participant j's decision at time t - see equations (20), (21)

$\lambda_j(t)$	=	M_j^h-vector with components $\lambda_{j,m}(t)$
$\tilde{\lambda}_j(\cdot,t)$	=	value of the multiplier $\lambda_j(t)$ which solves individual participants profit maximization problem - see equation (23)
$\hat{\lambda}_j(\cdot,t)$	=	value of the multiplier $\lambda_j(t)$ which solves global welfare maximization problem - see equation (45)
M_j^h	=	number of rows in the vector $h_j(\cdot,t)$
M_j^u	=	number of rows in the vector $u_j(t)$
M_j^x	=	number of rows in the vector $x_j(t)$
$NR_j(t)$	-	see equations (5) and (6)
P	=	number of industry products
P	=	$\{1,\ldots,P\}$, the set of industry products
$\theta_p(t)$	=	Kuhn-Tucker multiplier associated with balance of product p (i.e. equation (30)) - see (40)
$\theta(t)$	=	P-vector with components $\theta_p(t)$.
$\hat{\theta}(\cdot,t)$	=	value of the multiplier $\theta(t)$ which solves global welfare maximization problem - see equation (46)
$R_j[\cdot,t]$	=	profit for j during period t plus expected profit from t+1 to T+1 - see equations (7) and (16) and (25)
$\bar{r}_j(\cdot)$	-	see equation (15)
$\hat{r}_k^j(\cdot)$	-	see equation (56)
$\bar{\rho}_j(\cdot)$	-	see equation (17)
$S(t)$	-	see equations (26) and (27)
T	=	number of decision times
$\mathbf{T_d}$	=	$\{1,\ldots,T\}$, the set of decision times
$u_j(t)$	=	participant j's decision vector at decision time t, with components $u_{j,n}(t)$, $n = 1,\ldots,M_j^u$
$\tilde{u}_j(\cdot,t)$	=	decision vector which solves individual participants profit maximization problem - see equation (23)
$\hat{u}_j(\cdot,t)$	=	decision vector which solves global welfare maximization problem - see equation (44)
$\hat{U}^j(\cdot,t)$	-	see equation (54)
$W[\cdot,t]$	=	global welfare for the t^{th} period plus the expected global welfare for t+1 to T+1 - see equations (34),(39) and (47)
$\bar{\omega}(\cdot)$	-	see equation (38)
$x_j(t)$	=	state vector of participant j at time t

$\psi(\bullet,t)$ = optimal rule for choosing $\phi(t)$ - see equations (48) and (49)

$\bar{\psi}(\bullet,t+1)$ - see equation (53)

$\hat{\psi}_j[\bullet,t+1]$ see equation (55)

$y_{j,p}(\bullet)$ = nett consumption (less production) of product p by participant j during time interval t

$y_j(\bullet)$ = P-vector of $y_{j,p}(u_j(t),t)$, $p \in \mathbf{P}$

∇_j = $\frac{\partial}{\partial u_j(t)}$

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